ELEMENTARY AMENABLE GROUPS AND THE FARRELL-JONES CONJECTURE

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Abstract. In this note, we prove the K- and L-theoretical Farrell-Jones Conjecture with coefficients in an additive category for elementary amenable groups of finite Hirsch length.

Introduction

The class of elementary amenable groups is the class of groups generated from finite groups and $\mathbb{Z}$ by the operations of extension and increasing union. This class arose first in connection with the Banach-Tarski paradox. All virtually solvable groups are elementary amenable. The Farrell-Jones Conjecture is known for virtually solvable groups by recent work of C. Wegner ([9]). It is natural for one to ask whether the Farrell-Jones Conjecture holds for all elementary amenable groups. The Houghton's groups provide interesting examples of elementary amenable groups with finite Hirsch length that are not virtually solvable. Notice that elementary amenable groups with finite Hirsch length are locally finite by virtually solvable ([6]). In this note, we prove the following,

Theorem A. *Elementary amenable groups with finite Hirsch length satisfy the K- and L-theoretical Farrell-Jones conjecture with finite wreath product and coefficients in an additive category.*

In [2], Bieri and Sach generalized the definition of the Houghton's group to the higher dimensions. Theses groups provide interesting examples of elementary amenable groups with infinite Hirsch length. One can also show that these groups satisfy the Farrell-Jones Conjecture with similar methods.

For more information about the Farrell-Jones Conjecture with finite wreath product and coefficients in an additive category we refer to [9, Section 2.3] . We will abbreviate the K- and L-theoretical Farrell-Jones Conjecture with finite wreath product and coefficients in an additive category by FJCw. Note that FJCw implies the Farrell-Jones Conjecture with coefficients in an additive category.

Date: October, 2015.

2010 *Mathematics Subject Classification.* Primary 18F25; Secondary 19A31, 19B28, 19G24.

Key words and phrases. Farrell-Jones Conjecture; K-theory of group rings; L-theory of group rings; Elementary Amenable groups.

1. Inheritance Properties and Results on FJCw

We list some inheritance properties and results on FJCw that we may need.

Proposition 1.1. *(1) If a group G satisfies FJCw, then any subgroup $H_1 \subset G$ and any finite index over group $H_2 \supset G$ satisfies FJCw.*
(2) If G_1 and G_2 satisfy FJCw, then the direct product $G_1 \times G_2$ and free product $G_1 * G_2$ satisfy FJCw.
(3) Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps). If each G_i satisfies FJCw, then the colimit $colim_{i \in I} G_i$ satisfies FJCw.
(4) Let $\phi : G \to Q$ be a group homomorphism. If Q, $Ker(\phi)$ and $\phi^{-1}(C)$ satisfy FJCw for every infinite cyclic subgroup $C < Q$ then G satisfies FJCw.
(5) CAT(0) groups satisfy FJCw.
(6) Virtually solvable groups satisfy FJCw.
(7) Fundamental groups of graphs of Abelian groups satisfy FJCw.

Proof of (1) - (4) can be found for example in [9, Section 2.3] . (5) is the main result of [1] and [8] noticing that CAT(0) groups are closed under finite wreath product. (6) is proved in [9]. (7) is proved in [5].

2. Elementary amenable groups with finite Hirsch length

In this section, we prove FJCw for elementary amenable groups with finite Hirsch length. A group is locally finite if every finitely generated subgroup of it is finite.

Lemma 2.1. *If FJCw holds for all locally finite by infinite cyclic groups, then it holds for elementary amenable groups with finite Hirsch length.*

Proof Note first that elementary amenable groups with finite Hirsch length are locally finite by virtually solvable ([6]). Let Γ be a locally finite by virtually solvable group, we have the following short exact sequence,

$$1 \longrightarrow L \longrightarrow \Gamma \xrightarrow{\phi} V \longrightarrow 1$$

Where L is a locally finite group, V is a virtually solvable group, ϕ is a surjection from Γ to V. Since FJCw is known for all virtually solvable groups (Proposition 1.1 (6)), hence by Proposition 1.1 (4), in order to prove FJCw for Γ, we only need to prove for any infinite cyclic subgroup C of Γ, $\phi^{-1}(C)$ satisfies FJCw. $\phi^{-1}(C)$ sits inside the following short exact sequence

$$1 \longrightarrow L \longrightarrow \phi^{-1}(C) \longrightarrow C \longrightarrow 1$$

Hence $\phi^{-1}(C)$ is locally finite by infinite cyclic. Therefore we proved our Lemma. □

Note that $\phi^{-1}(C)$ is a semidirect product of the form $L \rtimes C$ where C acts on L by conjugation.

Proposition 2.2. *Let L be a locally finite group, then any semidirect product $L \rtimes \mathbb{Z}$ satisfies FJCw.*

Proof We can assume L has countable many generators. Note if L is finitely generated, then L is finite. Hence $L \rtimes \mathbb{Z}$ is virtually cyclic and those groups satisfy FJCw automatically. We now assume L is infinitely generated, with generators $\{x_i\}_{1\leq i<\infty}$. Let L_n be the subgroup of L generated by $\{x_i\}_{1\leq i<n}$. The colimit of L_n is L. Denote the generator of $\mathbb{Z}$ in $L \rtimes \mathbb{Z}$ by c, c acts on L. Let $A_n = \{x \in L_n \mid c \cdot x \in L_n\}$. A_n is a subgroup of L_n. Denote the subgroup $c \cdot A_n$ of L_n by B_n. Denote the map from A_n to B_n mapping x to $c \cdot x$ by f_n. Let $\Gamma_n = L_n *_{f_n}$ be the HNN extension of L_n by the stable letter t_n conjugating A_n to B_n via f_n.

There is natural embedding of L_n into L_{n+1}, hence there is a natural map from Γ_n to Γ_{n+1} mapping t_n to t_{n+1}. On the other hand there exists a natural map from Γ_n to $L \rtimes \mathbb{Z}$ mapping t_n to c, noticing L_n naturally embeds in L. One sees easily now that the colimit of Γ_n is $L \rtimes \mathbb{Z}$.

Notice that Γ_n are HNN extensions of finite groups, hence they act on their Bass-Serre tree with finite stabilizers. Therefore Γ_n are CAT(0) groups (in fact they are virtually free groups), and they satisfy FJCw by Proposition 1.1 (5). Then by Proposition 1.1 (3), their colimit $L \rtimes \mathbb{Z}$ also satisfies FJCw. □

We completed the proof of Theorem A.

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